# A Simple Proof of Sharkovsky's Theorem

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**Abstract**

In this note, we present a simple directed graph proof of Sharkovsky's theorem.

## 1 Introduction.

Throughout this note, $I$ is a compact interval, and $f : I \to I$ is a continuous map. For each integer $n \ge 1$, let $f^n$ be defined by: $f^1 = f$ and $f^n = f \circ f^{n-1}$ when $n \ge 2$. For $y$ in $I$, we call the set $O_f(y) = \{\, f^k(y) \,\big|\, k \ge 0 \,\}$ the *orbit* of $y$ (under $f$) and call $y$ a *periodic point* of $f$ with least period $m$ (or a period-$m$ point of $f$) if $f^m(y) = y$ and $f^i(y) \ne y$ when $0 < i < m$. If $f(y) = y$, then we call $y$ a *fixed point* of $f$. It is clear that every $f$ of the type in question has fixed points.

For discrete dynamical systems defined by iterated interval maps, one of the most remarkable results is Sharkovsky's theorem [**5**], [**6**]. It states that, if $f$ has a period-$m$ point, then $f$ also has a period-$n$ point precisely when $m \prec n$ in the following Sharkovsky's ordering :

$$3 \prec 5 \prec 7 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

It is well known (see [**8**]) that the sufficiency of Sharkovsky's theorem can be derived from the following three statements: (a) if $f$ has a periodic point of least period greater than 2, then $f$ also has a periodic point of least period 2; (b) if $f$ has a periodic point of odd period $m \ge 3$, then $f$ also has a periodic point of least period $n$ for every integer $n$ such that $n \ge m+1$; (c) if $f$ has a periodic point of odd period $m \ge 3$, then $f$ also has periodic points of all even periods. The difficulty of proving the sufficiency of Sharkovsky's theorem lies in proving (c), where most proofs involve the structures of the so-called Štefan cycles [**1**], [**3**], [**7**]. In this note, we give a unified proof of (b) and (c) that does not involve Štefan cycles. We also give a different proof of (a) [**2**], [**8**]. For the sake of completeness, we end the paper with a proof of Sharkovsky's theorem.

In proving Sharkovsky's theorem, we need the following result [**3**, p.12].

**Lemma 1.** *Let $k, m, n$, and $s$ be positive integers. Then the following statements hold:*

(1) *If $y$ is a periodic point of $f$ with least period $m$, then it is a periodic point of $f^n$ with least period $m/(m,n)$, where $(m,n)$ is the greatest common divisor of $m$ and $n$.*

(2) *If $y$ is a periodic point of $f^n$ with least period $k$, then it is a periodic point of $f$ with least period $kn/s$, where $s$ divides $n$ and is relatively prime to $k$.*

# 2 A proof of (a).

We need the following result, which can also be used to show that $f$ has no period-2 point if and only if for each point $c$ of $I$ the iterates $f^n(c)$ converge to a fixed point of $f$ [**3**, p.121].

**Lemma 2.** *If $c$ and $d$ are points of $I$ such that $f(d) \leq c < d \leq f(c)$, then $f$ has a periodic point of least period 2.*

*Proof.* Write $I = [a, b]$. Let $w = \min\{c \leq x \leq d \,|\, f(x) = x\}$, and let $v$ be a point in $[c, w]$ with $f(v) = d$. Then, $f^2(v) = f(d) \leq c \leq v$. If $f$ has no fixed point in $[a, c]$, then it fixes no point of $[a, v]$. Since $f^2(a) \geq a$, it follows that $f$ has a periodic point with least period 2 in $[a, v]$. If $f$ has a fixed point in $[a, c]$, let $t = \max\{a \leq x < c \,|\, f(x) = x\}$. Then $f$ has no fixed point in $(t, v]$. Let $u$ be a point in $[t, c]$ with $f(u) = c$. Then $f^2(u) = f(c) \geq d > u$. Since $f^2(v) \leq v$, we infer that $f^2(y) = y$ for some $y$ in $[u, v]$. Because $f$ has no fixed point in $[u, v]$, $y$ is a periodic point of $f$ with least period 2. ■

**Proposition 3.** *If $f$ has a periodic point of least period $m$ larger than 2, then $f$ also has a periodic point of least period 2.*

*Proof.* Let $P = \{x_i \,|\, 1 \leq i \leq m\}$, with $x_1 < x_2 < \cdots < x_m$, be a period-$m$ orbit of $f$. Since $x_1 < f(x_1)$ and $f(x_m) < x_m$, there exists an integer $s$ satisfying $1 \leq s \leq m-1$ such that $x_s = \max\{x \in P \,|\, x < f(x)\}$. It is clear that $x_{s+1} \leq f(x_s)$ and $f(x_{s+1}) \leq x_s$. By Lemma 2, $f$ has a periodic point of least period 2. ■

# 3 A unified proof of (b) and (c).

If there are closed subintervals $J_0, J_1, \cdots, J_{n-1}, J_n$ of $I$ with $J_n = J_0$ such that $f(J_i) \supset J_{i+1}$ for $i = 0, 1, \cdots, n-1$, then we say that $J_0 J_1 \cdots J_{n-1} J_0$ is a *cycle of length $n$*. We require the following result.

**Lemma 4.** *If $J_0 J_1 J_2 \cdots J_{n-1} J_0$ is a cycle of length $n$, then there exists a periodic point $y$ of $f$ such that $f^i(y)$ belongs to $J_i$ for $i = 0, 1, \cdots, n-1$ and $f^n(y) = y$.*

We now give a simple unified proof of (b) and (c).

**Proposition 5.** *If $f$ has a periodic point of least period $m$ with $m \geq 3$ and odd, then $f$ has periodic points of all even periods. Furthermore, $f$ has a periodic point of least period $n$ for each integer $n$ with $n \geq m+1$.*

*Proof.* Let $P = \{\, x_i \,\big|\, 1 \leq i \leq m \,\}$, with $x_1 < x_2 < \cdots < x_m$, be a period-$m$ orbit of $f$. Let $x_s = \max\{\, x \in P \,\big|\, x < f(x) \,\}$. Then $x_{s+1} \leq f(x_s)$ and $f(x_{s+1}) \leq x_s$, so $f$ has a fixed point $z$ in $[x_s, x_{s+1}]$. Since $m$ is odd, for some integer $t$ such that $1 \leq t \leq m-1$ and $t \neq s$ the points $f(x_t)$ and $f(x_{t+1})$ lie on opposite sides of $z$. Thus $f([x_t, x_{t+1}]) \supset [x_s, x_{s+1}]$. For simplicity, we assume that $x_t < x_s$. If $x_{s+1} \leq x_t$, the proof is similar. Let $q$ be the smallest positive integer such that $f^q(x_s) \leq x_t$. Then $2 \leq q \leq m-1$.

First assume that $m = 3$. Without loss of generality, we assume that $f(x_1) = x_2$, $f(x_2) = x_3$, and $f(x_3) = x_1$. Let $J_0 = [x_1, x_2]$ and $J_1 = [x_2, x_3]$. For any $n \geq 2$, we can apply Lemma 4 to the cycle $J_0 J_1 J_1 \cdots J_1 J_0$ of length $n$ to obtain a period-$n$ point. Accordingly, if $f$ has a period-3 point, then $f$ has periodic points of all periods. Now assume that $m > 3$. Since $q$ is the smallest positive integer such that $f^q(x_s) \leq x_t$, $x_{t+1} \leq f^i(x_s)$ whenever $1 \leq i \leq q-1$. If $x_{t+1} \leq f^{q-1}(x_s) < x_s$, Lemma 4 applies to the cycle

$$[x_t, f^{q-1}(x_s)][f^{q-1}(x_s), z][f^{q-1}(x_s), z][x_t, f^{q-1}(x_s)]$$

and establishes the existence of a period-3 point of $f$. If $f^{q-1}(x_s) = x_{s+1}$, we can apply Lemma 4 to the cycle

$$[z, x_{s+1}][x_t, x_{t+1}][x_s, x_{s+1}][z, x_{s+1}]$$

to obtain a period-3 point of $f$.

We proceed assuming that $x_{s+1} < f^{q-1}(x_s)$. If $k = \min\{\ \ 1 \leq i \leq q-1 \ \big|\ f^{q-1}(x_s) \leq f^i(x_s) \,\}$, then $x_{t+1} \leq f^{k-1}(x_s) < f^{q-1}(x_s)$, so either $x_{s+1} \leq f^{k-1}(x_s) < f^{q-1}(x_s)$ or $x_{t+1} \leq f^{k-1}(x_s) \leq x_s$. If $x_{s+1} \leq f^{k-1}(x_s) < f^{q-1}(x_s)$, we can invoke Lemma 4 for the cycle

$$[f^{k-1}(x_s), f^{q-1}(x_s)][z, f^{k-1}(x_s)][z, f^{k-1}(x_s)][f^{k-1}(x_s), f^{q-1}(x_s)]$$

to obtain a period-3 point of $f$. If $x_{t+1} \leq f^{k-1}(x_s) \leq x_s$ $(< z < f^{q-1}(x_s))$, we choose $u$ in $[x_t, x_{t+1}]$ such that $f(u) = z$, pick $w$ in $[z, f^{q-1}(x_s)]$ with $f(w) = u$, and let $v$ in $[f^{k-1}(x_s), z]$ be a point such that $f(v) = w$. By applying Lemma 4 to the cycle $[u, v][z, w][u, v]$ and, for every *even* integer $n \geq 4$, to the cycle

$$[u, v]([z, w][v, z])^{\frac{n-2}{2}}[z, w][u, v]$$

(here $([z, w][v, z])^{\frac{n-2}{2}}$ represents $(n-2)/2$ copies of $[z, w][v, z]$) of length $n$, we conclude that $f$ has periodic points of all even periods. On the other hand, let $J_i = [z : f^i(x_s)]$ for $i = 0, 1, \cdots, q-1$, where $[a : b]$ denotes the closed interval with $a$ and $b$ as endpoints. For any $n \geq m+1$, we appeal to Lemma 4 to the cycle of length $n$ $\ J_0 J_1 \ \cdots \ J_{k-1} J_{q-1} [x_t, x_{t+1}] J \cdots J J_0$, where $J = [x_s, x_{s+1}]$, to confirm the existence of a period-$n$ point. ■

# 4 A proof of Sharkovsky's theorem.

We now combine (a), (b), (c), and Lemma 1 to prove Sharkovsky's theorem.

**Theorem 6 (Sharkovsky).** *Assume that $f : I \to I$ is a continuous map. If $f$ has a period-$m$ point, then $f$ also has a period-$n$ point precisely when $m \prec n$ in the Sharkovsky's ordering defined as in Section 1.*

*Proof.* By (b) and (c), we have $3 \prec 5 \prec 7 \prec \cdots \prec 2 \cdot 3$. If $f$ has period-$(2 \cdot m)$ points with $m \geq 3$ and odd, then $f^2$ has period-$m$ points. By (b), $f^2$ has period-$(m+2)$ points, which by Lemma 1(2) implies that $f$ has either period-$(m+2)$ points or period-$(2 \cdot (m+2))$ points. If $f$ has period-$(m+2)$ points, then by (b) $f$ also has period-$(2 \cdot (m+2))$ points. In either case, $f$ has period-$(2 \cdot (m+2))$ points. On the other hand, by (c) $f^2$ has period-6 points and hence, by Lemma 1(2), $f$ has period-$(2^2 \cdot 3)$ points. Now if $f$ has period-$(2^k \cdot m)$ points with $m \geq 3$ and odd and if $k \geq 2$, then by Lemma 1(1) $f^{2^{k-1}}$ has period-$(2 \cdot m)$ points. It follows from what we have just proved that $f^{2^{k-1}}$ has period-$(2 \cdot (m+2))$ points and period-$(2^2 \cdot 3)$ points. In view of Lemma 1(2), $f$ has period-$(2^k \cdot (m+2))$ points and period-$(2^{k+1} \cdot 3)$ points. Furthermore, because $f$ has period-$(2^k \cdot m)$ points, $f^{2^k}$ has period-$m$ points. By (b), $f^{2^k}$ has period-$2^n$ points as long as $2^n > m$, so by Lemma 1(2) $f$ has period-$(2^{k+n})$ points for all integers $n$ such that $2^n > m$. Finally, if $f$ has period-$2^i$ points for some integer $i \geq 2$, then $f^{2^{i-2}}$ has period-4 points. As a result of (a), $f^{2^{i-2}}$ has period-2 points, ensuring that $f$ has period-$2^{i-1}$ points. This proves the sufficiency of Sharkovsky's theorem.

For the converse, it suffices to assume that $I = [0,1]$. Let $T(x) = 1 - |2x - 1|$ be the tent map on $I$. Then for any $k \geq 1$ the equation $T^k(x) = x$ has exactly $2^k$ distinct solutions in $I$. It follows that $T$ has finitely many period-$k$ orbits. Among these period-$k$ orbits, let $P_k$ be one with the smallest diameter $\max P_k - \min P_k$. For any $x$ in $I$, let $T_k(x) = \min P_k$ if $T(x) \leq \min P_k$, $T_k(x) = \max P_k$ if $T(x) \geq \max P_k$, and $T_k(x) = T(x)$ if $\min P_k \leq T(x) \leq \max P_k$. It is then easy to see that $T_k$ has exactly one period-$k$ orbit, i.e., $P_k$, and no period-$j$ orbit for any $j$ with $j \prec k$ in the Sharkovsky's ordering (see also [**1**], pp. 32-34]). Now let $Q_3$ be any period-3 orbit of $T$ of minimal diameter. Then $[\min Q_3, \max Q_3]$ contains finitely many period-6 orbits of $T$. If $Q_6$ is one of smallest diameter, then $[\min Q_6, \max Q_6]$ contains finitely many period-12 orbits of $T$. We choose one, say $Q_{12}$, of minimal diameter and continue the process inductively. Let $q_0 = \sup\{\min Q_{2^n \cdot 3} \,|\, n \geq 0\}$ and $q_1 = \inf\{\max Q_{2^n \cdot 3} \,|\, n \geq 0\}$. Let $T_\infty(x) = q_0$ if $T(x) \leq q_0$, $T_\infty(x) = q_1$ if $T(x) \geq q_1$, and $T_\infty(x) = T(x)$ if $q_0 \leq T(x) \leq q_1$. Then it is easy to check that $T_\infty$ has periodic points of least period $2^n$ for each $n \geq 0$, but has no periodic points of any other periods. This establishes the other direction in Sharkovsky's theorem. ■

**Remark.** Our method can also be used to prove that if $f$ has a periodic point of odd period $m > 1$, but no periodic points of odd period strictly between 1 and $m$ then any periodic orbit of odd period $m$ must be a Štefan orbit (cf. [**4**]).

**Acknowledgments.** I would like to thank M. Misiurewicz, A. N. Sharkovsky, and the referee for many constructive suggestions that led to improvements in this note.

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